Shear Stress in Beams
9.1 Introduction

- For beams subjected to pure bending, only tension and compression normal stresses are developed in the flexural member.

- In most situations, loadings applied to a beam create nonuniform bending; that is, internal bending moments are accompanied by internal shear forces.

- As a consequence of nonuniform bending, shear stresses as well as normal stresses are produced in the beam.

- In this chapter, a method will be derived for determining the shear stresses produced by nonuniform bending.
9.2 Resultant Forces Produced by Bending Stresses

- Application of concentrated force on simply supported beam made of pieces of wood.
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\[ I_z = 33750000 \text{ mm}^4 \]
9.2 Resultant Forces Produced by Bending Stresses

\[ A_1 = (50 \text{ mm})(120 \text{ mm}) = 6000 \text{ mm}^2 \]

Resultant \( F_B = (1.0 \text{ N/mm}^2)(6000 \text{ mm}^2) + \frac{1}{2} (2.0 \text{ N/mm}^2)(6000 \text{ mm}^2) \]

\[ = 12000 \text{ N} = 12 \text{ kN} \]

Resultant \( F_C = (1.5 \text{ N/mm}^2)(6000 \text{ mm}^2) + \frac{1}{2} (3.0 \text{ N/mm}^2)(6000 \text{ mm}^2) \]

\[ = 18000 \text{ N} = 18 \text{ kN} \]

\[ \sum F_X = 18 \text{ kN} - 12 \text{ kN} = 6 \text{ kN} \quad \sum F_X \neq 0 \]

• Is member (1) of beam segment BC in equilibrium?
• How much additional force is required to satisfy equilibrium?
• Where must this additional force be located?
9.2 Resultant Forces Produced by Bending Stresses

- What lessons can be drawn from this simple investigation?
  - In those beam spans where the internal bending moment is not constant, the resultant forces acting on portions of the cross section will be unequal in magnitude.
  - Equilibrium of these portions can be satisfied only by an additional shear force that is developed internally in the beam.
Example 9.1

A beam segment is subjected to the internal bending moments shown. The cross-sectional dimensions of the beam are given.

(a) Sketch a side view of the beam segment, and plot the distribution of bending stresses acting at sections A and B. Indicate the magnitude of key bending stresses in the sketch.

(b) Determine the resultant forces acting in the x direction on area (2) at sections A and B, and show these resultant forces in the sketch.

(c) Is the specified area in equilibrium with respect to forces acting in the x direction? If not, determine the horizontal force required to satisfy equilibrium for the specified area and show the location and direction of this force in the sketch.
Example 9.1

\[
\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{367500 \text{ mm}^3}{10500 \text{ mm}^2} = 35.0 \text{ mm}
\]

<table>
<thead>
<tr>
<th>(i)</th>
<th>(A_i (\text{mm}^2))</th>
<th>(y_i (\text{mm}))</th>
<th>(y_i A_i (\text{mm}^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>3000</td>
<td>50</td>
<td>150000</td>
</tr>
<tr>
<td>(2)</td>
<td>4500</td>
<td>15</td>
<td>67500</td>
</tr>
<tr>
<td>(3)</td>
<td>3000</td>
<td>50</td>
<td>150000</td>
</tr>
<tr>
<td></td>
<td>10500</td>
<td></td>
<td>367500</td>
</tr>
</tbody>
</table>

| \(i\) | \(I_{ei} (\text{mm}^4)\) | \(|d_i| (\text{mm})\) | \(d_i^2 A_i (\text{mm}^4)\) | \(I_z (\text{mm}^4)\) |
|------|-----------------|----------------|---------------------|---------------------|
| (1)  | 2500000         | 15.0           | 675000              | 3175000            |
| (2)  | 337500          | 20.0           | 1800000             | 2137500            |
| (3)  | 2500000         | 15.0           | 675000              | 3175000            |

\[
8487500
\]
Example 9.1

\[ \sigma_x = - \frac{M_y}{I_z} = - \frac{(11 \text{ kN-m})(65 \text{ mm})(1000 \text{ N/kN})(1000 \text{ mm/m})}{8487500 \text{ mm}^4} \]
\[ = -84.2 \text{ MPa} = 84.2 \text{ MPa (C)} \]

\[ F_A = \frac{1}{2} (6.48 \text{ MPa} + 45.4 \text{ MPa})(4500 \text{ mm}^2) \]
\[ = 116730 \text{ N} = 116.7 \text{ kN} \]

\[ F_B = \frac{1}{2} (9.72 \text{ MPa} + 68.0 \text{ MPa})(4500 \text{ mm}^2) \]
\[ = 174870 \text{ N} = 174.9 \text{ kN} \]
Example 9.1

\[ \Sigma F_x = 174.9 \text{ kN} - 116.7 \text{ kN} = 58.2 \text{ kN} \neq 0 \]

\( F_H \) must act at the boundaries between areas (1) and (2), and between areas (2) and (3).

By symmetry, half of the horizontal force will act on each surface.

Since \( F_H \) acts along the vertical sides of area (2), it is termed a shear force.
MecMovie Example M9.1

Discussion of the horizontal shear force developed in a flexural member.
9.3 The Shear Stress Formula

- A method for determining the shear stresses produced in a prismatic beam made of a homogeneous linear-elastic material will be developed.
9.3 The Shear Stress Formula

The integral term in the last equation is the first moment of area $A'$ about the neutral axis of the cross section. This quantity will be designated $Q$.
9.3 The Shear Stress Formula

What is the significance of Equation (9.1)?

- If the internal bending moment in a beam is not constant (i.e., $\Delta M \neq 0$), then an internal horizontal shear force $F_H$ must exist at $y = y_1$ in order to satisfy equilibrium.

- Furthermore, note that the term $Q$ pertains expressly to area $A'$. Since the value of $Q$ changes with area $A'$, so too does $F_H$. In other words, at every value of $y$ possible within a cross section, the internal shear force $F_H$ required for equilibrium is unique.
9.3 The Shear Stress Formula

Let us apply Equation (9.1) to the problem discussed in Section 9.2.

\[ F_H = \frac{\Delta MQ}{I_z} \]  
(9.1)

The area \( A' \) pertinent to this problem is simply the area of member (1), the 50 mm by 120 mm board at the bottom of the cross section.

The first moment of area \( Q \) is computed from \( \int y \, dA' \). Let the width of member (1) be denoted by \( b \). Since this width is constant, the differential area can be conveniently expressed as \( dA' = b \, dy \).

\[ Q = \int_{y=-25}^{y=-75} b \, dy = b \frac{1}{2} [y^2]_{y=-25}^{y=-75} = 300000 \, mm^3 \]

\[ F_H = \frac{\Delta MQ}{I_z} = \frac{(675 \, kN \cdot mm)(300000 \, mm^3)}{33750000 \, mm^4} = 6 \, kN \]
9.3 The Shear Stress Formula

Equation (9.1) can be extended to define the shear stress produced in a beam subjected to nonuniform bending.

\[ F_H = \frac{\Delta MQ}{I_z} \]  \hspace{1cm} (9.1)

\[ \tau = \frac{VQ}{l_z t} \]  \hspace{1cm} (9.2)

\[ \tau_{H,\text{avg}} = \frac{F_H}{t \Delta x} = \frac{\Delta MQ}{t \Delta x I_z} = \frac{\Delta M}{\Delta x I_z t} \]

\[ \Delta x \rightarrow 0, \frac{\Delta M}{\Delta x} \]

\[ \tau_H = \frac{dM}{dx} \frac{Q}{l_z t} \]

\[ \frac{dM}{dx} = V \]

\[ \tau_H = \frac{VQ}{l_z t} \]
9.3 The Shear Stress Formula

\[ F_H = \frac{\Delta MQ}{I_z} \] \hspace{1cm} (9.1)

\[ \tau = \frac{VQ}{I_z t} \] \hspace{1cm} (9.2)

Since \( Q \) varies with area \( A' \), the value of \( \tau \) varies over the depth of the cross section.

At the upper and lower boundaries of the cross section (e.g., points a, c, d, and f), the value of \( Q \) is zero since area \( A' \) is zero.

The maximum value of \( Q \) occurs at the neutral axis of the cross section. Accordingly, the largest shear stress \( \tau \) is usually located at the neutral axis.

Therefore, the maximum horizontal shear stress \( \tau \) occurs at the y coordinate that has the largest value of \( Q/t \). Most often, the largest value of \( Q/t \) does occur at the neutral axis, but this is not necessarily the case.
9.3 The Shear Stress Formula

\[ \tau = \frac{VQ}{I_z t} \quad (9.2) \]

Although the stress given by Equation (9.2) is associated with a particular point in a beam, it is averaged across the thickness \( t \) and hence is accurate only if \( t \) is not too great.

For a rectangular section having a depth equal to twice the width, the maximum stress computed by methods that are more rigorous is about 3 percent greater than that given by Equation (9.2).

If the cross section is square, the error is about 12 percent.

If the width is four times the depth, the error is almost 100 percent!
MecMovie Example M9.2

M9.2 Derivation of shear stress formula

Consider a beam subjected to transverse loading.
9.4 The First Moment of Area $Q$

- The first moment of area term appears as the numerator in the definition of a centroid

\[
\bar{y} = \frac{\int_A y \, dA}{\int_A dA}
\]

\[
Q = \int_{A'} y \, dA' = \bar{y}' \int_{A'} dA' = \bar{y}' A'
\]

- $Q$ is the first moment of area of only portion $A'$ of the total cross-sectional area $A$

- $\bar{y}'$ is the distance from the neutral axis of the cross section to the centroid of area $A'$

- $Q$ for cross sections that consist of $i$ shapes can be calculated as the summation

\[
Q = \sum_i y_i A_i
\]
The area $A'$ begins at this cut line and extends away from the neutral axis to the free surface of the beam.

\[ A' = A_1 + A_2 + A_3 \]

\[ \bar{y}' = \frac{y_1 A_1 + y_2 A_2 + y_3 A_3}{A_1 + A_2 + A_3} \]

\[ Q = y' A' = \frac{y_1 A_1 + y_2 A_2 + y_3 A_3}{A_1 + A_2 + A_3} (A_1 + A_2 + A_3) = y_1 A_1 + y_2 A_2 + y_3 A_3 \]

\[ Q = \sum_{i} y_i A_i \]
9.5 Shear Stresses in Beams of Rectangular Cross Section

\[ Q = y' \Delta' = \left[ y + \frac{1}{2} \left( \frac{h}{2} - y \right) \right] \left( \frac{h}{2} - y \right) b = \frac{1}{2} \left( \frac{h^2}{4} - y^2 \right) b \]

\[ \tau = \frac{VQ}{I_z t} = \frac{V}{\left( \frac{1}{12} bh^3 \right) b} \times \frac{1}{2} \left( \frac{h^2}{4} - y^2 \right) b = \frac{6V}{bh^3} \left( \frac{h^2}{4} - y^2 \right) \]

\[ \tau_{\max} = \frac{VQ}{I_z t} = \frac{V}{\left( \frac{1}{12} bh^3 \right) b} \times \frac{1}{2} \left( \frac{h}{2} \right) bh = \frac{3V}{2bh} = \frac{3V}{2A} \]
9.5 Shear Stresses in Beams of Rectangular Cross Section

The shear stress intensity associated with an internal shear force $V$ in a rectangular cross section is distributed parabolically in the direction perpendicular to the neutral axis (i.e., in the $y$ direction) and uniformly in the direction parallel to the neutral axis (i.e., in the $z$ direction).

The shear stress vanishes at the upper and lower edges of the rectangular cross section and peaks at the neutral axis location.

It is important to remember that shear stress acts on both transverse and longitudinal planes.
Example 9.2
A 3.2 m long simply supported laminated wood beam consists of eight 40 mm by 180 mm planks glued together to form a section 180 mm wide by 320 mm deep, as shown. The beam carries a 45 kN concentrated load at midspan. At section a–a located 0.8 m from support A, determine
(a) the average shear stress in the glue joints at b, c, and d.
(b) the maximum average shear stress in the cross section.
Example 9.2

From the shear-force diagram, the internal shear force $V$ acting at section a–a is

$V = 22.5 \text{ kN}$.
Example 9.2

\[ I_z = \frac{(180 \text{ mm})(320 \text{ mm})^3}{12} = 491.520 \times 10^6 \text{ mm}^4 \]

\[ \tau = \frac{VQ}{I_z t} \]

\[ Q_b = (180 \text{ mm})(40 \text{ mm})(140 \text{ mm}) = 1.008 \times 10^6 \text{ mm}^3 \]

\[ \tau_b = \frac{VQ_b}{I_z t_b} = \frac{(22.5 \text{ kN})(1.008 \times 10^6 \text{ mm}^3)(1000 \text{ N/kN})}{(491.520 \times 10^6 \text{ mm}^4)(180 \text{ mm})} \]

\[ = 0.256 \text{ MPa} = 256 \text{ kPa} \]
Example 9.2

\[ Q_c = (180 \text{ mm})(120 \text{ mm})(100 \text{ mm}) = 2.160 \times 10^6 \text{ mm}^3 \]

\[ \tau_c = \frac{VQ_c}{I_z t_c} = \frac{(22.5 \text{ kN})(2.160 \times 10^6 \text{ mm}^3)(1000 \text{ N/kN})}{(491.520 \times 10^6 \text{ mm}^4)(180 \text{ mm})} \]

\[ = 0.549 \text{ MPa} = 549 \text{ kPa} \]

\[ Q_d = (180 \text{ mm})(80 \text{ mm})(120 \text{ mm}) = 1.728 \times 10^6 \text{ mm}^3 \]

\[ \tau_d = \frac{VQ_d}{I_z t_d} = \frac{(22.5 \text{ kN})(1.728 \times 10^6 \text{ mm}^3)(1000 \text{ N/kN})}{(491.520 \times 10^6 \text{ mm}^4)(180 \text{ mm})} \]

\[ = 0.439 \text{ MPa} = 439 \text{ kPa} \]
Example 9.2

\[ Q_{\text{max}} = (180 \text{ mm})(160 \text{ mm})(80 \text{ mm}) = 2.304 \times 10^6 \text{ mm}^3 \]

\[ \tau_{\text{max}} = \frac{VQ_{\text{max}}}{I_z t} = \frac{(22.5 \text{ kN})(2.304 \times 10^6 \text{ mm}^3)(1000 \text{ N/kN})}{(491.520 \times 10^6 \text{ mm}^4)(180 \text{ mm})} \]

\[ = 0.586 \text{ MPa} = 586 \text{ kPa} \]
9.6 Shear Stresses in Beams of Circular Cross Section

\[ A' = \frac{\pi r^2}{2} \]

\[ \bar{y}' = \frac{4r}{3\pi} \]

\[ Q = \bar{y}'A' = \frac{4r}{3\pi} \frac{\pi r^2}{2} = \frac{2}{3}r^3 \]

\[ Q = \frac{1}{12}d^3 \]

\[ \tau_{\max} = \frac{VQ}{I_zt} = \frac{V}{\frac{\pi r^4}{4}} \times \frac{2}{3} \frac{r^3}{2r} \times \frac{1}{2r} = \frac{4V}{3\pi r^2} = \frac{4V}{3A} \]
9.7 Shear Stresses in Webs of Flanged Beams

To determine the shear stress at a point a located in the web of the cross section, the calculation for Q consists of determining the first moment of the two highlighted areas (1) and (2) about the neutral axis z.

A substantial portion of the total area of a flanged shape is concentrated in the flanges, so the first moment of area (1) about the z axis makes up a large percentage of Q.

While Q increases as the value of y decreases, the change is not as pronounced in a flanged shape as it would be for a rectangular cross section.
9.7 Shear Stresses in Webs of Flanged Beams

Consequently, the distribution of shear stress magnitudes over the depth of the web, while still parabolic, is relatively uniform.

The minimum horizontal shear stress occurs at the junction between the web and the flange, and the maximum horizontal shear stress occurs at the neutral axis.

For wide-flange steel beams, the difference between the maximum and minimum web shear stresses is typically in the range of 10–60 percent.
In deriving the shear stress formula, it was assumed that the shear stress across the width of the beam (i.e., in the z direction) could be considered constant.

This assumption is not valid for the flanges of beams; therefore, shear stresses computed for the top and bottom flanges from Equation (9.2) and plotted in Figure 9.13a are fictitious.

Shear stresses are developed in the flanges (1) of a wide-flange beam, but they act in the x and z directions, not the x and y directions.
Homework for Chapter 9:

MecMovies Examples and Exercises
M9.2, M9.3, M9.5, M9.6 - M9.8